

MAS 117

Introduction to Algebra

Solutions 9

March 2007

1 Take any two elements a and b . We have $a^2 = a$, $b^2 = b$ and $(a+b)^2 = a+b$. So

$$\begin{aligned}(a+b)^2 &= a+b, \\ a^2 + ab + ba + b^2 &= a+b, \\ a + ab + ba + b &= a+b, \\ ab + ba &= 0,\end{aligned}$$

the last line using the Cancellation Law. So $ba = -ab$ for all a, b . Putting $b = a$, we see that $a^2 = -a^2$, that is, $a = -a$, for all a . So $-ab$ is the same as ab and we have $ba = ab$ for all a, b ; that is, R is commutative. Also, as $a = -a$, we have $a + a = 0$ for all $a \in R$.

A simpler argument to show that $a = -a$ is to observe that $a^2 = (-a)^2$ in any ring, while in our case $a^2 = a$ and $(-a)^2 = -a$.

2 If it is a group, we have to verify all four axioms. If not, we only have to show that one axiom fails!

(a) Not a group. It is the inverse law (G3) that fails.

First we have to work out which element is the identity. Which set E satisfies $E \cap X = X$ for all subsets X ? This equation says $X \subseteq E$, so E must be the largest possible subset, namely S itself.

Now there is no set X such that $\emptyset \cap X = S$, so $\emptyset$ has no inverse.

(b) This is a group:

Closure: The composition of two rotations is a rotation.

Associative law: If we rotate through angles α , β and γ , the result is a rotation through $\alpha + \beta + \gamma$, no matter how we bracket the terms.

Identity: The rotation through 0 is the identity.

Inverse: The inverse of a rotation through α is a rotation through $-\alpha$.

(c) This is a group:

Closure: The sum of two even numbers is even.

Associative law: This holds because it holds in general for integer addition.

Identity: 0 is an even integer.

Inverse: The inverse of $2m$ is $-2m$ which is even.

You can also (and more easily) do this by showing that it is a subgroup of the additive group of $\mathbb{Z}$, as we did for $4\mathbb{Z}$ in lectures.

(d) This is not a group. The identity law (G2) fails: there is no *even* integer e such that $ex = x$ for all even integers x .

(e) This is not a group. The associative law (G1) fails. For

$$(2 - 3) - 4 = -5, \quad 2 - (3 - 4) = 1.$$

3 From the given information, we have the following part of the Cayley table:

$\circ$	e	a	b	c
e	e	a	b	c
a	a	b		
b	b			
c	c			

Now imagine you are solving a Sudoku puzzle. What can occur in row a and column c ? It cannot be a or b (which already occur in that row) or c (which already occurs in that column); so it must be e . Then the only element left to go in row a is c (in column b).

Similarly we find e in row c and column a , and c in row b and column a .

Then row b , column c must have entry a (the only entry not occurring in this row so far), and the rest of the table is easy to complete:

$\circ$	e	a	b	c
e	e	a	b	c
a	a	b	c	e
b	b	c	e	a
c	c	e	a	b

Why is this argument valid? In other words, why can a Cayley table not have the same element more than once in a row or column? The reason is the *cancellation laws*. Suppose, for example, that the element x occurred twice in row g , so that, say, $g \circ h = x = g \circ k$. By the cancellation law, it follows that $h = k$. (Remember that this is proved by multiplying the equation $g \circ h = g \circ k$ on the left by g^{-1} .) Similarly the other cancellation law shows that we cannot have repeats in a column.